

Weekly Assignment 1 (Solutions)

Overview: Below are some possible solutions to the exercises from Weekly 1. Keep in mind that there may be many ways to solve each problem. Your grade on Weekly 1 will be determined by the grader using the [Weekly Assignment Rubric](#).

Solutions:

1. Exercise [9.1.15](#)

- Solution.* a. This trace describes how the volume V of 1 mole of an ideal gas is related to the temperature T if the pressure P is fixed at $P = 1000$ pascals. The trace is the curve described by the equation $z = V(1000, T) = 0.008314T$. Since this curve is a line, we can be very specific: the trace tells us that when the pressure is fixed at $P = 1000$ pascals, the volume V of 1 mole of a gas increases linearly with the temperature T at a rate of 0.008314 cubic meters per unit Kelvin.
- b. This trace describes how the volume of 1 mole of a gas is related to the pressure P if the temperature is fixed at $T = 5$ Kelvin. The trace is the hyperbola described by the equation $z = 41.57/P$. This tells us that when the temperature is fixed at $T = 5$ Kelvin, the volume is inversely related to the pressure P . More specifically, if the temperature is fixed at $T = 5$ Kelvin, then the volume of 1 mole of a gas decreases as the pressure increases, and conversely.
- c. The level curve with $V = 0.5$ describes the relationship between the temperature T and the pressure P when the volume of 1 mole of a gas is fixed at $V = 0.5$ cubic meters. As long as the pressure is non-zero (meaning we are not in a perfect vacuum), the equation of the curve is $P = 16.628T$. This tells us that when the volume is fixed at $P = 0.5$ cubic meters, the pressure P increases linearly with the temperature T at a rate of 16.628 pascal per Kelvin.
- d. You should have drawn a picture. I will just graph it: [GeoGebra: Ideal Gas Law](#). Note that temperature and pressure are nonnegative by definition, therefore the domain of this function is $D = \{(P, T) : 0 \leq T, 0 \leq P\}$.

□

2. Exercise [9.1.17](#)

- Solution.* a. The domain is the set of all points for which $4 - x^2 - y^2 \geq 0$, since the square root of a negative number is not a real number. More precisely, the domain D is given by

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}.$$

This is a closed disk in the xy -plane centered at the origin with radius 4: [GeoGebra: A Disk](#).

- b. Any point (x, y) in D satisfies $0 \leq x^2 + y^2 \leq 4$. Therefore, the range is the set

$$R = \{8 - \sqrt{t} \in \mathbb{R} : t \in [0, 4]\} = [6, 8].$$

- c. The traces are circles, I used $c = 6.25, 6.5, 6.75, 7$ and got this: [GeoGebra: Exercise 9.1.17.c](#).
- d. I used $c = -1, 0, 1, 1.5, 1.75$ and got this: [Exercise 9.1.17.d](#).

- e. The traces are exactly the same curves (just in different directions on the graph).
- f. Using just the traces, you can conclude that the graph is a circular bowl shape centered on the z axis and opening in the positive z direction. Using the range you can determine where the top and the bottom of the bowl are. Here's the graph: [Exercise 9.1.17 graph](#).

□

3. Exercise 9.2.16

Solution. a. The vectors $\mathbf{u}$ and $\mathbf{v}$ form right triangles with the horizontal. The hypotenuses of the triangles are $|\mathbf{u}|$ and $|\mathbf{v}|$, respectively. Using trigonometry, it is clear that

$$\mathbf{u} = -|\mathbf{u}| \cos(60)\mathbf{i} + |\mathbf{u}| \sin(60)\mathbf{j} = -\frac{1}{2}|\mathbf{u}|\mathbf{i} + \frac{\sqrt{3}}{2}|\mathbf{u}|\mathbf{j}$$

and

$$\mathbf{v} = |\mathbf{v}| \cos(45)\mathbf{i} + |\mathbf{v}| \sin(45)\mathbf{j} = \frac{\sqrt{2}}{2}|\mathbf{v}|\mathbf{i} + \frac{\sqrt{2}}{2}|\mathbf{v}|\mathbf{j}.$$

- b. We are given that the magnitude of the force of gravity $\mathbf{w}$ acting to pull the picture downwards is 50 pounds. Since we assumed the wires meet at the origin, we can take $\mathbf{w} = -50\mathbf{j}$.
- c. The picture will hang in equilibrium if $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$. Comparing components in this vector equation, we get a system of equations

$$\begin{cases} 0 &= -\frac{1}{2}|\mathbf{u}| + \frac{\sqrt{2}}{2}|\mathbf{v}| \\ 50 &= \frac{\sqrt{3}}{2}|\mathbf{u}| + \frac{\sqrt{2}}{2}|\mathbf{v}|. \end{cases}$$

Solve by elimination to get $|\mathbf{u}| = \frac{100}{\sqrt{3}+1}$ and $|\mathbf{v}| = \frac{100}{\sqrt{6}+\sqrt{2}}$. Plug back in to the expressions from part (a) to complete the solution.

□

4. Exercise 9.3.11

Solution. a. The vectors are not perpendicular because $\langle 2, -1 \rangle \cdot \mathbf{v} = 2(-2) + (-1)5 = -9$.

- b. Suppose that $\mathbf{u} = \langle a, b \rangle$ is a unit vector so that $1 = |\mathbf{u}|^2 = a^2 + b^2$. Further assume that $\mathbf{u}$ is perpendicular to $\mathbf{v} = \langle -2, 5 \rangle$ so that $0 = \mathbf{u} \cdot \mathbf{v} = -2a + 5b$. Now, we have a system of two equations that we can solve by substitution. We have $a = \frac{5}{2}b$ so that $1 = \frac{29}{4}b^2$. The second equation has precisely two solutions, $b = \pm \frac{2}{\sqrt{29}}$. So there are precisely two vectors that fit the bill, $\mathbf{u} = \pm \langle \frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}} \rangle$. This makes sense geometrically.
- c. It is not, because the dot product is not zero.
- d. The method to solve this part is similar to part (b). But in this case, there are infinitely many unit vectors perpendicular to $\mathbf{y}$. The set of terminal points of all possible vectors traces out a circle in $\mathbb{R}^3$ with radius equal to 1. Here's a visualization: [GeoGebra: All Unit Vector Perp. to a Fixed Vector](#).
- e. Suppose that $\mathbf{r} = \langle a, b, c \rangle$ is a unit vector perpendicular to both $\mathbf{y}$ and $\mathbf{z}$. Then we get a system of equations

$$\begin{cases} 1 &= a^2 + b^2 + c^2 \\ 0 &= 2a + b \\ 0 &= 3b - 2c. \end{cases}$$

By substituting the second two equations into the first, we get $1 = a^2 + 4a^2 + 9a^2 = 14a^2$ so that $a = \pm \frac{1}{\sqrt{14}}$. So there are two unit vectors with the desired property, namely $\mathbf{r} = \pm \langle \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \rangle$. If you ignore the condition that $\mathbf{r}$ should have unit length, then the set of all terminal points of vectors perpendicular to both $\mathbf{y}$ and $\mathbf{z}$ is a line!

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5. Exercise 9.3.14

Solution.

- a. Using the definition of the dot product, we have

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \langle 1, 2, -1 \rangle \cdot \langle 8, 4, 8 \rangle = 8$$

and

$$(\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w}) = \langle 1, 2, -1 \rangle \cdot \langle 4, -3, 6 \rangle + \langle 1, 2, -1 \rangle \cdot \langle 4, 7, 2 \rangle = -8 + 16 = 8.$$

I notice that they are equal.

- b. There are two possible proofs. The first uses the properties of the dot product.

Proof. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be vectors in $\mathbb{R}^n$. Using [Properties of the Dot Product](#) we have

$$\begin{aligned} \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) &= (\mathbf{y} + \mathbf{z}) \cdot \mathbf{x} && (1. \text{ Commutative Property}) \\ &= (\mathbf{y} \cdot \mathbf{x}) + (\mathbf{z} \cdot \mathbf{x}) && (2. \text{ Distributive Property}) \\ &= (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z}) && (1. \text{ Commutative Property}) \end{aligned}$$

This proves the claim. Notice how this property is a consequence of properties 1 and 2 of the dot product - we did not need to appeal to the definition of the dot product at all to prove the claim.

The second proof uses the definition of the dot product.

Proof. Let $\mathbf{x} = \langle x_1, \dots, x_n \rangle$, $\mathbf{y} = \langle y_1, \dots, y_n \rangle$ and $\mathbf{z} = \langle z_1, \dots, z_n \rangle$. Then

$$\begin{aligned} \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) &= \sum_{i=1}^n x_i(y_i + z_i) \\ &= \sum_{i=1}^n x_i y_i + \sum_{i=1}^n x_i z_i \\ &= (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z}). \end{aligned}$$

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